# GLOBALIZATION FOR PARTIAL $H$-BIMODULE ALGEBRAS

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### Abstract

It will be defined a globalization for a partial $H$-bimodule algebra, extending the notion given by Alves and Batista in [3]. It will be shown that every partial $H$-bimodule algebra has a globalization, it will be constructed the standard one and showed that it is minimal.

## Bimodule algebra

An algebra $B$ is a left $H$-module algebra if there exist a linear map $\triangleright : H \otimes B \to B$, denoted by $\triangleright(h \otimes a) = h \triangleright a$, such that:

**(i)** $1_H \triangleright b = b$;
**(ii)** $h \triangleright (ab) = (h_1 \triangleright a)(h_2 \triangleright b)$;
**(iii)** $h \triangleright (g \triangleright b) = hg \triangleright b$;

In a similar way we can define a right $H$-module algebra.
We will say that $B$ is a $H$-bimodule algebra, if $B$ is a left and right $H$-module algebra such that the actions are compatible, i.e.,

$$h \triangleright (b \triangleleft k) = (h \triangleright b) \triangleleft k$$

## Examples

**Example 1:** Any algebra $B$ is an $H$-bimodule algebra with the trivial structure given by $\varepsilon_H$, i.e.,

$$h \triangleright a = \varepsilon(h)a \text{ and } a \triangleleft h = \varepsilon(h)a$$

**Example2:** A Hopf algebra $H$ is a $H^*$-bimodule algebra with the classical structure, i.e.

$$f \rightharpoonup h = h_1 f(h_2) \text{ and } h \leftharpoonup f = f(h_1)h_2$$

## Partial bimodule algebra

An algebra $A$ is a left partial $H$-module algebra if there exists a linear map $\to: H \otimes A \to A$, denoted by $\to (h \otimes a) = h \to a$, such that:

**(i)** $1_H \to a = a$;
**(ii)** $h \to [a(g \to b)] = (h_1 \to a)(h_2 g \to b)$;

In a similar way we can define a right partial $H$-module algebra.
We will say that $A$ is a partial $H$-bimodule algebra, if $A$ is a left and right partial $H$-module algebra such that the corresponding partial actions are compatible, i.e.,

$$h \to (b \leftarrow k) = (h \to b) \leftarrow k$$

## Examples

**Example 1:** Let $\mathbb{H}_4 = \langle x, g | \ x^2 = 0, g^2 = 1, gx = -xg \rangle$ be the Sweedler algebra and $A$ any $\mathbb{K}$-algebra. So $A$ is a partial $\mathbb{H}_4$-bimodule algebra by the following actions:

$$\begin{array}{rcl} 1 \to a = a & \qquad & a \leftarrow 1 = a \\ g \to a = 0 & & a \leftarrow g = 0 \\ x \to a = la & & a \leftarrow x = ra \\ xg \to a = -la & & a \leftarrow xg = -ra \end{array},$$

$\forall a \in A$ and any $r, l \in \mathbb{K}$.
Note that, it is not a global action.

**Example 2:** Let $G$ be a finite group and $G_1, G_2$ subgroups of $G$ such that $car(\mathbb{K})$ does not divide their respective orders $|G_1|$ and $G_2$. Let $\mathbb{K}G^*$ be the dual algebra of $\mathbb{K}G$ with basis $\{p_g | \ g \in G\}$ and $A$ a $\mathbb{K}$-algebra.
So $A$ is an $\mathbb{K}G^*$-bimodule algebra with actions defined by:

$$p_g \to a = \begin{cases} \frac{1}{|G_1|}a, \text{ if } g \in G_1 \\ 0, \text{ otherwise} \end{cases} \quad \text{and} \quad a \leftarrow p_g = \begin{cases} \frac{1}{|G_2|}a, \text{ if } g \in G_2 \\ 0, \text{ otherwise.} \end{cases}$$

Note that each above action is global if and only if the corresponding subgroup is equal to 1.

## Induced Partial Action

Let $B$ be an $H$-bimodule algebra with actions denoted by $\triangleleft$ and $\triangleright$, $A$ a unital subalgebra of $B$ such that $\forall a, b \in A$

$$(a \triangleleft h)(k \triangleright b) = (a \triangleleft h)1_A(k \triangleright b)$$

in $A$.
So we define the following linear maps

$$\begin{aligned} \to: H \otimes A &\to A \\ h \otimes a &\mapsto h \to a = 1_A(h \triangleright a) \end{aligned}$$

$$\begin{aligned} \leftarrow: A \otimes H &\to A \\ a \otimes h &\mapsto a \leftarrow h = (a \triangleleft h)1_A \end{aligned}$$

With these maps $A$ becomes a partial $H$-bimodule algebra.

**Remark:** Let $B$ be a left $H$-module algebra. With the right trivial structure given by $\varepsilon$, $B$ is a $H$-bimodule algebra. In this context, the induced partial action as bimodule is the same as the induced as left partial action. The induced action for left $H$-module algebra was defined by Alves and Batista in [3].

## Globalization

Let $A$ a partial $H$-bimodule algebra, with partial actions $\leftarrow$ and $\to$. A pair $(B, \theta)$, where $B$ is an $H$-bimodule algebra and $\theta : A \to B$ is a multiplicative monomorphism, is said a globalization for the partial $H$-bimodule algebra if:

**(i)** $(\theta(a) \triangleleft h)(k \triangleright \theta(b)) = \theta[(a \leftarrow h)(k \to b)]$;
**(ii)** $B$ is the $H$-bimodule generated by $\theta(A)$, i.e., $B = H \triangleright \theta(A) \triangleleft H$.

## Remark

**(1)** In the above definition, $\theta(A)$ is a partial $H$-bimodule algebra by the partial action induced from $B$. Note that, these induced partial actions are equivalent to the corresponding partial actions of $A$, i.e.,

$$\theta(h \to a) = h \to \theta(a) \text{ and } \theta(a \leftarrow h) = \theta(a) \leftarrow h.$$

**(2)** If $(B, \theta)$ is a globalization for $A$, so the multiplication of $B$ is given by

$$(h \triangleright \theta(a) \triangleleft k)(h' \triangleright \theta(a') \triangleleft k') = h_1 \triangleright \theta[(a \leftarrow kS(k'_1))(S(h_2)h' \to a')] \triangleleft k'_2.$$

**(3)** If $(B, \theta)$ is a globalization for a partial $H$-bimodule algebra $A$, so $(H \triangleright \theta(A), \theta)$ is a globalization for $A$ as partial left $H$-module algebra and $(\theta(A) \triangleleft H, \theta)$ is a globalization for $A$ as partial right $H$-module algebra.

## Standard Globalization

Let $A$ be a partial $H$-bimodule algebra and consider $\text{Hom}(H \otimes H, A)$, which is an algebra with the convolution product.
Define

$$\begin{aligned} \triangleright : H \otimes \text{Hom}(H \otimes H, A) &\to \text{Hom}(H \otimes H, A) \\ h \otimes f &\mapsto (x \otimes y \mapsto f(xh \otimes y)) \\ \triangleleft : \text{Hom}(H \otimes H, A) \otimes H &\to \text{Hom}(H \otimes H, A) \\ f \otimes h &\mapsto (x \otimes y \mapsto f(x \otimes hy)) \end{aligned}$$

**Proposition:**

Let $A$ be a partial $H$-bimodule algebra, so $\text{Hom}(H \otimes H, A)$ with the above structure is an $H$-bimodule algebra.

Now consider the multiplicative monomorphism

$$\begin{aligned} \varphi : A &\to \text{Hom}(H \otimes H, A) \\ a &\mapsto (h \otimes k \mapsto h \to a \leftarrow k) \end{aligned}$$

Note that the condition

$$(\varphi(a) \triangleleft h) * (k \triangleright \varphi(b)) = \varphi((a \leftarrow h)(k \to b))$$

in the definition of globalization is trivially satisfied.
With this construction, we have the following theorem.

**Theorem:**
Every partial $H$-bimodule algebra has a globalization.

The globalization above constructed is called the *standard* globalization.

## Theorem

Let $(B', \theta)$ a globalization for the $H$-bimodule algebra $A$. Then there exists an algebra epimorphism $\Phi$ from $(B', \theta)$ onto $(B, \varphi)$,
The above morphism is defined by

$$\begin{aligned} \Phi : B' &\to B \\ h \triangleright \theta(a) \triangleleft k &\mapsto h \triangleright \varphi(a) \triangleleft k. \end{aligned}$$

## Minimal Globalization

A globalization $(B, \theta)$ is called minimal if for all $H$-subbimodule $M$ of $B$ satisfying $\theta(1_A)M\theta(1_A) = 0$ we have $M = 0$.

Note that, the standard globalization is minimal.
Moreover, when a globalization is minimal we have that $\Phi$ is an algebra isomorphism.

## References

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